Another method of deriving auto-Backlund transformations for nonlinear evolution equations

1983 J. Phys. A: Math. Gen. 16 891

(http://iopscience.iop.org/0305-4470/16/4/024)

J. Phys. A: Math. Gen. 16 (1983) 891–892. Printed in Great Britain

COMMENT

Another method of deriving auto-Bäcklund transformations for nonlinear evolution equations

Huang Xun-Cheng
Shanghai Institute of Computing Technique, 30 Hu Nan Road, Shanghai, China

Received 23 July 1982, in final form 14 September 1982

Abstract. In this comment, we show that a Bäcklund transformation between two equations one of which has an auto-Bäcklund transformation implies an auto-Bäcklund transformation for the second which can thus be found by simple algebraic operations.

The ordinary methods to derive Bäcklund transformations of equations are the methods obtained by Clairin, Chen and Hirota (Miura 1976, Bullough and Caudrey 1980). In this comment, we show that a Bäcklund transformation between two equations one of which has an auto-Bäcklund transformation implies an auto-Bäcklund transformation for the second which can be found by simple algebraic operations. The following two equations are considered as examples for that purpose.

As is well known, there is a Bäcklund transformation

$$\phi_x = -(2\nu)^{-1}u\phi, \qquad \phi_t = -(4\nu)^{-1}(2\nu u_x - u^2)\phi, \tag{1a, b}$$

between the Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0 \qquad (\nu > 0) \tag{2}$$

which is a typical dissipative wave equation, and the diffusion equation

$$\phi_t = \nu\phi_{xx} \qquad (\nu > 0) \tag{3}$$

which has an auto-Bäcklund transformation

$$\phi' - \sqrt{\nu}\phi_x = 0, \qquad \sqrt{\nu}\phi'_x - \phi_t = 0, \tag{4a, b}$$

where ϕ and ϕ' are two solutions of equation (3).

From (1) and (4) we have proved that equation (2) has the auto-Bäcklund transformation

$$u_x = (2\nu)^{-1}u(u-u'), \qquad u_t = 2^{-1}u(u-u')_x - 2^{-1}u_x(u+u'), \tag{5a, b}$$

where u and u' are two solutions of equation (2).

We can use the same method to consider the generalised sine-Gordon equation

$$z_{xt} = 2a(z_x z_t)^{1/2} \qquad (a = \text{constant}) \tag{6}$$

and the linear Klein equation

$$\psi_{xt} = a^2\psi \qquad (a = \text{constant}) \tag{7}$$

which has an auto-Bäcklund transformation

$$\psi'_x = \psi_x + ka(\psi' - \psi), \qquad \psi'_t = \psi_t + k^{-1}a(\psi' - \psi), \qquad (8a, b)$$

where ψ and ψ' are two solutions of equation (7) and k is any non-zero parameter.

Using a Bäcklund transformation (Dodd and Bullough 1976),

$$z_x = a^{-2}\psi_x^2, \qquad z_t = \psi^2, \qquad (9a, b)$$

between equations (6) and (7), we have obtained an auto-Bäcklund transformation for equation (6), which is defined as

$$z'_{tt}(z_t)^{1/2} - z_{tt}(z'_t)^{1/2} = 2k^{-1}(z'_t(z_t)^{1/2} - z_t(z'_t)^{1/2}), \qquad (10a)$$

$$(z'_x)^{1/2} - (z_x)^{1/2} = k((z'_t)^{1/2} - (z_t)^{1/2}), \qquad (10b)$$

where z and z' are two solutions of equation (6) and k is a non-zero arbitrary parameter.

References

Bullough R K and Caudrey P J (ed) 1980 *Solitons, Topics in Current Physics* vol 17 (Berlin: Springer)
Dodd R K and Bullough R K 1976 *Proc. R. Soc.* A **351** 499–523
Miura R M (ed) 1976 *Bäcklund Transformations, Lecture Notes in Mathematics* vol 515 (Berlin: Springer)